Visualizing Euclidean Rhythms Using Tangle Theory

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Abstract

Recently there has been growing interest in music theory to develop and study advanced geometrical models for the analysis of both pitch space and isomorphic rhythmic structures. Toussaint recently discovered ways of generating several well-known cyclical rhythms that have defined the rhythmic structure of many traditional musics of West Africa, Cuba, Brazil and beyond. By generating rhythms from a binary format described by Bjorklund using the Euclidean algorithm, Toussaint has described a combinatorial classification for these rhythms by representing their unique interval vectors on a two-dimensional circular lattice. This visual representation can be useful for both the analysis and algorithmic generation of cyclical rhythms. To encourage further research and to explore the role of image schemas within musical rhythm, the authors find a relationship between the generative process of Euclidean rhythms and their connection to geometrical representations unique to tangle theory.

1. Motivation

The evenness and structural stability of musical scales, pitch class sets, and rhythms has long been of interest to music theorists. Recently there has been an increasing interest in the geometric and numerical properties of rhythmic design, including the pictorial representation to expand the number of analytical tools for studying their structure. Toussaint discovered and described in detail the structural properties of Euclidean rhythms—cyclical rhythmic patterns that spread their pulses out as evenly as possible. By using a taxonomy of well-defined rhythmic structures, we can explore a visualization method for the benefit of understanding rhythm in a new way.

The evenness and deepness of Euclidean rhythms are of interest to many mathematicians and music theorists primarily because they are a class of rhythms that appear in a diverse range of world musics. Using Euclidean rhythms as a point of departure, the authors take advantage of the continued fraction for converting Euclidean rhythms to a visual domain using tangle theory. The nearly identical generative process for generating rhythms and generating tangles can lead to a unique understanding of Euclidean rhythms as well as produce the basis
for a new class of image schemas. Image schemas are informed by visual representations and patterns abstracted from bodily experience—in this case we have used the tying of knots. Through Conway’s notation of tangles within knots, a 2D visual and conceptual model can be created that maps onto the abstract domain of cyclical rhythm. While Brower describes ways in which image schemas play an important role in our embodied understanding of tonal music, they can also be applied to the structure and analysis of rhythm. Image schemas are structures that organize mental representations—they are usually diagrammed in abstract ways helping us to have a better visual understanding of a musical metaphor. Mapping image schemas onto geometrical rhythmic timelines reflect many of Lakoff and Johnson’s ideas about conceptual metaphors in general. For example the conceptual source domain of a tangle within a knot can be paired with the target conceptual domain of the cyclic and periodic nature of a rhythmic ostinato. This correspondence might demonstrate the larger idea that musical thinking can be metaphorical in that it involves mapping of patterns from one domain of experience to another.

Using not only the 2D visual model of tangles, but also the step process of generating tangles, we can more easily visualize metaphoric mappings that involve the process of tying knots. Brower has introduced ways in which we can use embodied image schemas that can, among other things, help relate musical ideas to basic features of bodily experience such as space, time, force, and motion. Approaches to image schemas, conceptual metaphors, and visualizations of abstract space are not only important in the analysis of musical structures, but have been researched for the purpose of creating new musical interfaces. While many familiar visual representations of rhythmic space are used to study timing and performance, there are others that can be used specifically to reflect its organization, measure its similarity and to represent schematics for composition. Like rhythm, the tying (or visualization) of knots makes the metaphor of goal-directed motion vivid. The patterns within the class of rhythms designated as Euclidean rhythms may demonstrate what Larson describes as musical inertia—the tendency of a pattern to continue in the same fashion. Separately our experience of musical motion (rhythm) borrows aspects of physical motion that can be mapped onto musical succession, including the constraints of physical forces. The construction of tangles within knots both reflects the inertia of Euclidean rhythm structure and the cyclical looping process of their physical performance. Physical motions tend to have beginnings, middles and ends that move from stability through instability then back again to stability. Furthermore tangles can provide an insight not just for composers and theorists, but for the design of musical interfaces for the generation of rhythmic patterns.
2. Euclidean Rhythms

Using Bjorklund’s sequence generation algorithm, Toussaint describes how the Euclidean algorithm generates a class of well-known cyclical rhythms, denoted $E(n,m)$. As noted earlier these rhythms are best understood by the distribution of onsets evenly within a set total of pulses. For example the algorithm for $E(5,8)$ distributes the five onsets among eight pulses as evenly as possible. While seemingly simple in construction the family of Euclidean rhythms represent a wide range of temporal organizations. Much like the rhythmic structures of Central African traditional music, Euclidean rhythms have the ability to obtain complex, subtle, varied and strictly coherent systems, with a limited number of elements.

2.1 Generating Rhythms

Many short rhythmic patterns described here as Euclidean rhythms have appeared in practice in numerous music cultures in the world. The elegance of this generative algorithm is reflected in its simple musical design. Using the visual programming language for audio and multimedia, Max/MSP, a simple program can be created to generate both the onsets and offsets of any given Euclidean rhythm (Fig. 1).

![Figure 1. Euclidean rhythm generator in Max/MSP](image-url)
The program interface shown in Figure 1 simply computes the greatest common divisor of two given integers represented by the number of total pulses and the desired number of onsets within a given ostinato. Digital audio examples of the rhythms described in this article, along with other examples of Euclidean rhythms can be heard and downloaded at an archive created by the authors.\textsuperscript{10}

2.2 Visualizing Rhythm
Euclidean rhythms can be represented on a circular timeline to better visualize the inter-onset interval patterns and rotations of a given rhythm.\textsuperscript{10} In addition to the circular timeline there are many non-conventional circular graphing methods designed for the study of cyclical rhythm. One purpose for relying on circular or tactus-based rhythmic diagrams is that the rhythm’s organization will help us understand how it is heard.\textsuperscript{11} In addition Benadon convincingly argues that geometrical thinking serves the music composition process as made evident by the large number of hand-drawn schematics used by composers throughout the centuries. Finding a visualization method that generates a unique correlate for each instance of a unique rhythm is appealing from both a compositional and analytical perspective.

3. Continued Fractions

3.1 Musical Analysis Using Continued Fractions
Both finite and infinite continued fractions have important use in the theory of Pythagorean scales, well-formed scales, tuning, and temperament. They can be helpful both in categorizing hierarchies of well-formed scales, but also in drawing musical analogues to visual phenomena such as Penrose tilings and Ammann-Beenker tilings.\textsuperscript{12} Here, the revealing nature of continued fractions that provide, say, the description and implementation of scales and unique temperaments may also provide a tactile process for generating Euclidean rhythms: the twisting of knots.
3.2 Notation
Here, \(a_0\) is any integer and the remaining \(a_i\) are positive integers.

\[
[a_0; a_1, a_2, ..., a_n] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\ldots + \frac{1}{a_n}}}}}
\]

The continued fraction representation of real numbers is not unique, but it is finite if and only if the real number is rational. Since Euclidean rhythms are generated from rational numbers, we will not need to consider infinite continued fractions. Moreover, we need only consider two potential representations for any rational number.

3.3 Theorem
Every rational number can be represented in precisely two ways as a finite continued fraction,

\[
[a_0; a_1, a_2, ..., a_n] = [a_0; a_1, a_2, ..., a_{n-1}, a_n + 1],
\]

where the latter is called the canonical continued fraction representation of a rational number.\(^{13}\) Because continued fractions offer us an alternative to representing numbers as a decimal, their application in music theory is important when studying the structures and applications of musical ratios in general. Besides their importance in understanding various topics within tuning and temperament, their application in the study of cross and polyrhythms can be quite useful.

4. Tangles

A tangle is a proper embedding of the disjoint union of two arcs into a 3-ball, easily represented two dimensionally (2D). An example of this 2D representation using Conway’s notation is seen within its circular context (Figure 2).
Tangles are considered topologically while keeping the ends of the arcs fixed. In other words, inside the 3-ball, the strands of the arc can be continuously deformed, so long as the arc never passes through itself (and hence violating the continuity of the deformation). The simplest type of tangles are called rational tangles. First introduced by British mathematician John Conway, these tangles are constructed from two horizontal arcs via a series of twists. These twists offer us a unique way of understanding the tying of the knot.

4.1 Conway’s Notation
Conway provided a notation for representing rational tangles that mirrors that of continued fractions and provides the foundation for their relationship with Euclidean rhythms. We illustrate this notation in Figure 3 with the construction of the tangle (2,3,4). Starting with two horizontal strands, the tangle (0), twist horizontally by rotating the right side of the 3-ball counterclockwise. A single 180° counterclockwise rotation about the horizontal axis yields the tangle (1), and two such twists creates the tangle (2). Next, twist the bottom of the 3-ball three times counterclockwise (around the vertical axis), followed by four horizontal twists to create the tangle (2,3,4).
Note that in the third step the tangle is denoted (2,3,0). Mathematical convention is to terminate the iterative process with horizontal twists. Thus, the length of any tangle will be always be odd. Also note that in the previous example, all rotations were counterclockwise. Had any of the rotations been in the clockwise direction, the tangle notation would be denoted using a negative integer. Since tangles are topological objects, it is possible that two rational tangles may actually be the same (called isotopic). For example, the tangle (-2,3,2) can be continuously deformed into the tangle (3,-2,3). This is resolved through continued fractions.

4.2 Tangle Fractions
To any rational tangle there is a natural associated continued fraction and corresponding theorem,

\[(a_0, a_1, \ldots, a_n) \Leftrightarrow a_n + \frac{1}{a_{n-1} + \frac{1}{a_{n-2} + \ldots + \frac{1}{a_0}}}
\]

Conway’s theorem states that two rational tangles are isotopic if and only if their continued fractions are equal.\(^{15}\) Therefore a rational number corresponds to an isotopy class of rational tangles, yielding the one-to-one correspondence between Euclidean rhythms and tangle representations of them. We can quite easily assign a tangle to any and all Euclidean rhythms.

\[[a_0; a_1, \ldots, a_n] \Leftrightarrow (a_n, a_{n-1}, \ldots, a_0)\]

We conclude with two examples of this correspondence (Figure 4 and Figure 5).
4.3. Rhythm Example 1, E(5,8) Cinquillo

\[
\frac{8}{5} = [1;1,1,2] \Leftrightarrow (2,1,1,1,0)
\]

Figure 4. Tangle representation of Cuban Cinquillo rhythm

4.4. Rhythm Example 2, E(5,13)

\[
\frac{13}{5} = [2;1,1,2] \Leftrightarrow (2,1,1,2,0)
\]

Figure 5. Tangle representation of Euclidean rhythm E(5,13)

5. CONCLUSIONS

5.1 Euclidean Rhythms
In this paper we have shown the connection between the class of musical rhythms known as Euclidean rhythms and tangle theory for the purpose of creating a unique visual analogue for any unique rhythm within that class. Through Conway’s notation, the resulting 2D representations of any Euclidean rhythm can serve as a rendering of image-schematic structures, which gives coherence and structure to our musical experience. The twisting of tangles within knots can also
represent the source-path-goal schema that is now so frequently applied to our understanding of musical structures.

5.2 Future Work
Because every unique Euclidean rhythm has a unique 2D tangle representation, we are aware of the possibility that tangle theory might be used to analyze this class of rhythms on a variety of levels. One such level is to measure both the similarity and the objective complexity of Euclidean rhythms while taking into account the fact that the duration of these rhythms can vary infinitely in length. Toussaint describes numerous geometric features for measuring similarity, including viewing rhythms as sequences of symbols. Tangle theory could be used to aid techniques of pattern matching and pattern recognition to further shed light on how visualization and analysis of rhythm informs our understanding of musical meaning.

Notes

6. Ibid.


16. Lakoff & Johnson, *Metaphors We Live By*.


18. Ibid.